Global Journal of Pure and Applied Mathematics (GJPAM) ISSN 0973-1768 Volume 12,Number 4 (2016)

Redefined T-fuzzy right h-ideals of hemirings

*G. Mohanraj**
Department of Mathematics,
Annamalai University,
Tamil Nadu, India.
Email: gmohanraaj@gmail.com

E. Prabu
Department of Mathematics,
Annamalai University,
Tamil Nadu, India.
Email: ersprabumath@gmail.com

***Abstract*— We redefine the concepts of T-fuzzy right ideals and T-fuzzy right h-ideals of hemirings by using T-sum and T-product. We establish various necessary and sufficient conditions for a fuzzy set to be a T-fuzzy right ideal and T-fuzzy right h-ideal. The concept of T-fuzzy h-closure is introduced. We generalize the notion of h-closure into T-fuzzy h-closure.**

Keywords*— Hemirings, T-norm, T-fuzzy right ideals, T-fuzzy right h-ideals

I. INTRODUCTION

In 1965, the origin of fuzzy sets was introduced by L.A.Zadeh [14]. Latter it was applied in group theory by Rosenfeld [13]. Since then, many authors further introduced fuzzy sub-semigroups, fuzzy sub-rings, fuzzy sub-semirings, fuzzy sub-hemirings, fuzzy ideals and fuzzy sub-algebras, and so on (see, [7, 2, 5]). The notion of h-ideals in hemirings was initiated by D.R.La Torre [9] in 1965. The general properties of fuzzy h-ideals of hemirings were described in [12, 7, 15]. P.Dheena and G.Mohanraj [1] introduced T-fuzzy bi-ideal and T -fuzzy quasi ideal in a ring.

In this paper, the notions of T-fuzzy right ideals and T-fuzzy right h-ideals of hemirings are redefined by using T-sum and T-product. We establish various necessary and sufficient conditions for a fuzzy set to be a T-fuzzy right ideal and T-fuzzy right h-ideal. The concept of T-fuzzy h-closure is introduced. We generalize the notion of h-closure into T-fuzzy h-closure.

Basic definitions and mathematical facts about triangular norms can be found in [8].

II. PRELIMINARIES

Definition 2.1.[16] *An algebraic system* $(S,+,\cdot)$ *is called a semiring if* $(S,+)$ *and* $(S,\cdot)$ *are semigroups, that follows both distributive laws:*

$$x\cdot(y+z)=x\cdot y+x\cdot z \text{ and } (x+y)\cdot z=x\cdot z+y\cdot z$$

for all $x,y,z\in S$.

Definition 2.2. *An additively commutative semiring S is called a hemiring if it has an absorbing element* $0\in S$ *such that*

$$0\cdot a=0=a\cdot 0 \text{ and } 0+a=a=a+0$$

for all $a\in S$.

A hemiring $(S,+,\cdot)$ *in which "$\cdot$" is commutative is called commutative hemiring.*

Definition 2.3. *[1]* A triangular norm [T-norm] is a binary operation T on [0,1], such that for all $x,y,z\in[0,1]$ which satisfies the following conditions:

i) $T(x,y)=T(y,x)$

ii) $T(x,T(y,z))=T(T(x,y),z)$

iii) If $x\geq x^*$ *and* $y\geq y^*$ *then* $T(x,y)\geq T(x^*,y^*)$

iv) $T(x,1)=T(1,x)=x$

Note: *Some basic triangular norms [8] are defined as follows:*

i) Minimum T-norm: $T_M(x,y)=\min\{x,y\}$

ii) Product T-norm: $T_P(x,y)=x\cdot y$

iii) Lukasiewicz T-norm: $T_L(x,y)=\max\{x+y-1,0\}$

iv) Drastic product T-norm:

$T_D(x,y)=0$ *if* $x,y\in[0,1)$ *and*

$T_D(x,y)=\min\{x,y\}$ *is otherwise,*

(v) Hamacher T-norms: for any $\lambda\in[0,\infty]$

$$(T_\lambda^H)(x,y)=\begin{cases}\mathrm{T_D(x,y)} & \text{if } \lambda=\infty\\ 0 & \text{if } \lambda=\mathrm{x}=\mathrm{y}=0\\ \dfrac{\mathrm{xy}}{\lambda+(1-\lambda)(\mathrm{x}+\mathrm{y}-\mathrm{xy})} & \textit{otherwise}\end{cases}$$

Definition 2.4. *A fuzzy set* η *of a hemiring S is a mapping* $\eta:S\to[0,1]$.

Definition 2.5. *A non-empty subset A of a hemiring S is called a right [left] ideal in S if (A,+) is closed and* $AS\subseteq A(SA\subseteq A)$.

Definition 2.6. *A right [left] ideal A of a hemiring S is called a right [left] h-ideal if* $a_1,b_1\in A$ *and* $x_1+a_1+z_1=b_1+z_1$ *imply* $x_1\in A$ *for* $x_1,z_1\in S$.

Definition 2.7. *Let B be a subset of a hemiring S. The h-closure of B, denoted* $\overline{B}$ *is defined as:*

$$\overline{B}=\{a\in S\mid a+b_1+z=b_1'+z\} \text{ for } b_1,b_1'\in B,\ z\in S\}.$$

---

*Corresponding author.

This paper was presented by the second author in the National Conference on AMASE-2016 conducted in Department of Mathematics, University College of Engineering Pattukkottai, Thanjavur, Tamil Nadu, India, on 22nd January 2016.

Definition 2.8. *Let η and δ be the fuzzy sets of a hemiring S and T be a triangular norm on [0, 1]. The T-sum of fuzzy sets η and δ is defined as follows:*

$$(\eta +_T \delta)(p) = \bigvee_{p=q+r} T(\eta(q), \delta(r))$$

Remark: Instead of T-norm, if we take minimum T-norm in Definition 2.8, the T-sum is referred to as a sum of fuzzy sets η and δ.

Definition 2.9. *Let η and δ be the fuzzy sets of a hemiring S and T be a triangular norm on [0, 1]. The T-product of fuzzy sets η and δ is defined as follows:*

$$(\eta \cdot_T \delta)(p) = \begin{cases} \bigvee_{p=qr} T(\eta(q), \delta(r)) & \text{if } p = qr \\ 0 & \text{if p cannot be expressible as } p = qr \end{cases}$$

Remark: Instead of T-norm, if we take minimum T-norm in Definition 2.9, the T-product is called a fuzzy product of fuzzy sets η and δ.

III. Redefined T-fuzzy right h-ideals

Throughout this paper, unless otherwise specified, S denotes a hemiring, T indicates a triangular norm on [0,1] and "1" is a fuzzy set on S defined as 1(p) = 1 for all $p \in S$.

Definition 3.1. *A fuzzy set η of S is called a T-fuzzy right [left] ideal of S if*

i) $\eta(p+q) \geq T(\eta(p), \eta(q))$

ii) $\eta(pq) \geq \eta(p)$ $[\eta(pq) \geq \eta(q)]$ *for all* $p, q \in S$.

Definition 3.2. *A fuzzy set η of S is called a fuzzy right [left] ideal of S if*

i) $\eta(p+q) \geq \min\{\eta(p), \eta(q)\}$

ii) $\eta(pq) \geq \eta(p)$ $[\eta(pq) \geq \eta(q)]$ *for all* $p, q \in S$.

Lemma 3.3. *Let η be a fuzzy set of S. Then the following conditions are equivalent*

i) $\eta(p+q) \geq T(\eta(p), \eta(q))$ *for all* $p, q \in S$.

ii) $\eta +_T \eta \subseteq \eta$.

Proof. Let $\eta(p+q) \geq T(\eta(p), \eta(q))$ for all $p, q \in S$. If $p = q + r$, then $\eta(p) = \eta(q+r) \geq T(\eta(q), \eta(r))$. Thus

$$\eta(p) \geq \bigvee_{p=q+r} T(\eta(q), \eta(r))$$
$$= (\eta +_T \eta)(p).$$

Therefore $\eta(p) \geq (\eta +_T \eta)(p)$ for all $p \in S$ implies $\eta +_T \eta \subseteq \eta$.

Conversely, let $\eta +_T \eta \subseteq \eta$ implies $\eta(p) \geq (\eta +_T \eta)(p)$. Thus

$$\eta(p+q) \geq (\eta +_T \eta)(p+q)$$
$$= \bigvee_{p+q=a+b} T(\eta(a), \eta(b))$$
$$\geq T(\eta(p), \eta(q)).$$

Therefore $\eta(p+q) \geq T(\eta(p), \eta(q))$ for all $p, q \in S$.

Theorem 3.4. *A fuzzy set η of S is a T-fuzzy right [left] ideal of S if and only if*

i) $\eta +_T \eta \subseteq \eta$

ii) $\eta \cdot_T 1 \subseteq \eta [1 \cdot_T \eta \subseteq \eta]$

Proof. Let η be a T-fuzzy right ideal of S. By Lemma 3.3, $\eta +_T \eta \subseteq \eta$. If p cannot be expressible as $p = qr$, then $(\eta \cdot_T 1)(p) = 0$. Therefore $0 = (\eta \cdot_T 1)(p) \leq \eta(p)$. If $p = qr$, then $\eta(p) = \eta(qr) \geq \eta(q) = T(\eta(q), 1(r))$. Thus

$$\eta(p) \geq \bigvee_{p=qr} T(\eta(q), 1(r))$$
$$= (\eta \cdot_T 1)(p).$$

Therefore $\eta(p) \geq (\eta \cdot_T 1)(p)$ for all $p \in S$. Hence $\eta \cdot_T 1 \subseteq \eta$. Similarly, we prove that if η is a T-fuzzy left ideal of S, then $1 \cdot_T \eta \subseteq \eta$.

Conversely, by Lemma 3.3, $\eta(p+q) \geq T(\eta(p), \eta(q))$ for all $p, q \in S$. Now, $\eta \cdot_T 1 \subseteq \eta$ implies $\eta(p) \geq (\eta \cdot_T 1)(p)$. Thus

$$\eta(pq) \geq (\eta \cdot_T 1)(pq)$$
$$= \bigvee_{pq=ab} T(\eta(a), 1(b))$$
$$\geq T(\eta(p), 1(q))$$
$$= \eta(p)$$

Therefore $\eta(pq) \geq \eta(p)$ for all $p \in S$. Hence η is a T-fuzzy right ideal of S. Similarly, we prove that if $1 \cdot_T \eta \subseteq \eta$, then η is a T-fuzzy left ideal of S,

Corollary 3.5. *A fuzzy set η of S is a fuzzy right [left] ideal of S if and only if*

i) $\eta + \eta \subseteq \eta$

ii) $\eta \cdot 1 \subseteq \eta$ $[1 \cdot \eta \subseteq \eta]$

Proof. By taking $T(a,b) = \min\{a, b\}$ for all $a, b \in S$ in Theorem 3.4, we get the result.

Definition 3.6. *Let η be a fuzzy set of S. A T-fuzzy right [left] ideal of S is called a T-fuzzy right [left] h-ideal of S if* $x_1 + a_1 + z_1 = b_1 + z_1$ *implies* $\eta(x_1) \geq T(\eta(a_1), \eta(b_1))$ *for* $x_1, a_1, b_1, z_1 \in S$.

Definition 3.7. *Let η be a fuzzy set of S. A fuzzy right [left] ideal of S is called a fuzzy right [left] h-ideal of S if*

$x_1 + a_1 + z_1 = b_1 + z_1$ *implies* $\eta(x_1) \geq \min\{\eta(a_1), \eta(b_1)\}$ *for* $x_1, a_1, b_1, z_1 \in S$.

Definition 3.8. *Let* η *be a fuzzy set of S. Then the T-fuzzy h-closure denoted by* $\overline{\eta}_T$ *of S is defined as:*

$$(\overline{\eta}_T)(x_1) = \bigvee_{x_1+a_1+z_1=b_1+z_1} T(\eta(a_1), \eta(b_1)) \text{ for } x_1, a_1, b_1, z_1 \in S.$$

Remark: Instead of T-norm, if we take minimum T-norm in Definition 3.8, $\overline{\eta}_T$ denoted by $\overline{\eta}$ is called a fuzzy h-closure of η of S.

Theorem 3.9. *A T-fuzzy right [left] ideal* η *of S is T-fuzzy right [left] h-ideal if and only if* $\overline{\eta}_T \subseteq \eta$.

Proof. Let η be a T-fuzzy right h-ideal of S. Now, $x_1 + a_1 + z_1 = b_1 + z_1$ implies $\eta(x_1) \geq T(\eta(a_1), \eta(b_1))$. Thus

$$\eta(x_1) = \bigvee_{x_1+a_1+z_1=b_1+z_1} T(\eta(a_1), \eta(b_1)) = \overline{\eta}_T(x_1).$$

Therefore $\overline{\eta}_T \subseteq \eta$.

Conversely, let $\overline{\eta}_T \subseteq \eta$ implies $\eta(x_1) \geq \overline{\eta}_T(x_1)$ for all $x_1 \in S$. Thus if $x_1 + a_1 + z_1 = b_1 + z_1$, then

$$\eta(x_1) \geq \overline{\eta}_T(x_1) \geq \bigvee_{x_1+a_1+z_1=b_1+z_1} T(\eta(a_1), \eta(b_1)) \geq T(\eta(a_1), \eta(b_1)).$$

Therefore η is a T-fuzzy right h-ideal of S. Similarly, we prove that a T-fuzzy left ideal η of S is T-fuzzy left h-ideal of S if and only if $\overline{\eta}_T \subseteq \eta$.

Theorem 3.10. *A fuzzy set* η *of S is a T-fuzzy right [left] h-ideal of S if and only if*
i) $\eta +_T \eta \subseteq \eta$,
ii) $\eta \cdot_T 1 \subseteq \eta [1 \cdot_T \eta \subseteq \eta]$,
iii) $\overline{\eta}_T \subseteq \eta$.

Proof. The proof follows from the Theorem 3.4 and Theorem 3.9.

Corollary 3.11. *A fuzzy set* η *of S is a fuzzy right [left] h-ideal of S if and only if*
i) $\eta + \eta \subseteq \eta$,
ii) $\eta \cdot 1 \subseteq \eta \ [1 \cdot \eta \subseteq \eta]$,
iii) $\overline{\eta} \subseteq \eta$.

Proof. By taking $T(a,b) = \min\{a,b\}$ for all $a, b \in S$ in Theorem 3.10, we get the result.

Theorem 3.12. *Every fuzzy right [left] h-ideal is a T-fuzzy right [left] h-ideal of S for any T-norm.*

Proof. Let η be fuzzy right h-ideal of S. For any T-norm, $T(a,b) \leq T_m(a,b) = \min\{a,b\}$. Thus, for any T-norm $\eta(p+q) \geq \min\{\eta(p), \eta(q)\} \geq T(\eta(p), \eta(q))$ and $\eta(pq) \geq \eta(p)$. Now $x_1 + a_1 + z_1 = b_1 + z_1$ implies $\eta(x_1) \geq \min\{\eta(a_1), \eta(b_1)\} \geq T(\eta(a_1), \eta(b_1))$. Therefore η is a T-fuzzy right h-ideal of S for any T-norm. Similarly, we prove that every fuzzy left h-ideal is a T-fuzzy left h-ideal of S for any T-norm.

Remark: Converse of Theorem 3.12 need not be true as shown by the following Example 3.13.

Example 3.13. *Let* $S = \{0, a_1, a_2, a_3\}$ *be a hemiring by the Cayley table as follows:*

$+$	0	a_1	a_2	a_3
0	0	a_1	a_2	a_3
a_1	a_1	0	a_3	a_2
a_2	a_2	a_3	0	a_1
a_3	a_3	a_2	a_1	0

$\cdot$	0	a_1	a_2	a_3
0	0	0	0	0
a_1	0	0	0	0
a_2	0	a_1	a_2	a_3
a_3	0	a_1	a_2	a_3

We define a fuzzy set η as follows:

$$\eta(x) = \begin{cases} 0.6 & if \ \ x = a_3 \\ 0.5 & if \ \ x = 0 \\ 0.45 & if \ \ x = a_1 \\ 0.4 & if \ \ x = a_2 \end{cases}$$

Clearly, η is a T_P-fuzzy right h-ideal of S for the product T-norm. But

$$\eta(a_3 + a_3) = \eta(0) = 0.5 \ngeq 0.6 = \min\{\eta(a_3), \eta(a_3)\}$$

implies η is not fuzzy right h-ideal of S.

Definition 3.14. *Let* η *and* δ *be the fuzzy sets of a hemiring S and T be a triangular norm on [0, 1]. A T-intersection* η *and* δ *denoted by* $T(\eta, \delta)$ *on S is defined as follows:*

$$T(\eta, \delta)(p) = T(\eta(p), \delta(p))$$

for all $p \in S$.

Remark: Instead of T-norm, if we take minimum T-norm in Definition 3.14, T-intersection is known as a intersection of fuzzy sets η and δ.

Theorem 3.15. *Let* η *and* δ *be the fuzzy sets of S. If* η *and* δ *are T-fuzzy right [left] ideals of S, then* $T(\eta, \delta)$ *is a T-fuzzy right [left] ideal of S.*

Proof. Let η and δ be the T-fuzzy right ideals of S and let $p, q \in S$. Now

$$T(\eta, \delta)(p+q) = T(\eta(p+q), \delta(p+q))$$

$$\begin{aligned}&\geq T(T(\eta(p),\eta(q)),T(\delta(p),\delta(q)))\\&=T(\eta(p),T(\eta(q),T(\delta(p),\delta(q))))\\&=T(\eta(p),T(T(\eta(q),\delta(p)),\delta(q)))\\&=T(\eta(p),T(T(\delta(p),\eta(q)),\delta(q)))\\&=T(\eta(p),T(\delta(p),T(\eta(q),\delta(q))))\\&=T(T(\eta(p),\delta(p)),T(\eta(q),\delta(q)))\\&=T(T(\eta,\delta)(p),T(\eta,\delta)(q)).\end{aligned}$$

Therefore $T(\eta,\delta)(p+q)\geq T(T(\eta,\delta)(p),T(\eta,\delta)(q))$ for all $p,q\in S$. Now

$$\begin{aligned}T(\eta,\delta)(pq)&=T(\eta(pq),\delta(pq))\\&\geq T(T(\eta(p),\delta(p)))\\&=T(T(\eta,\delta)(p)).\end{aligned}$$

Therefore $T(\eta,\delta)(pq)\geq T(T(\eta,\delta)(p))$ for all $p,q\in S$. Hence $T(\eta,\delta)$ is a T-fuzzy right ideal of S, Similarly, we prove that if η and δ are T-fuzzy left ideals of S, then $T(\eta,\delta)$ is a T-fuzzy left ideal of S.

Corollary 3.16. *Let η and δ be the fuzzy sets of S. If η and δ are fuzzy right [left] ideals of S, then $\eta\cap\delta$ is a fuzzy right [left] ideal of S.*

Proof. Instead of T-norm, if we take minimum T-norm in Theorem 3.15, we get the result.

Theorem 3.17. *Let η and δ be the fuzzy sets of S. If η and δ are T-fuzzy right [left] h-ideals of S, then $T(\eta,\delta)$ is a T-fuzzy right [left] h-ideal of S.*

Proof. Let η and δ be the T-fuzzy right h-ideals of S. By Theorem 3.15, $T(\eta,\delta)$ is a T-fuzzy right ideal of S. Now $x_1+a_1+z_1=b_1+z_1$ and $x_1\in S$ implies

$\delta(x_1)\geq T(\delta(a_1),\delta(b_1))$ and $\eta(x_1)\geq T(\eta(a_1),\eta(b_1))$.
Then

$$\begin{aligned}T(\eta,\delta)(x_1)&=T(\eta(x_1),\delta(x_1))\\&\geq T(T(\eta(a_1),\eta(b_1)),T(\delta(a_1),\delta(b_1)))\\&=T(\eta(a_1),T(\eta(b_1),T(\delta(a_1),\delta(b_1))))\\&=T(\eta(a_1),T(T(\eta(b_1),\delta(a_1)),\delta(b_1)))\\&=T(\eta(a_1),T(T(\delta(a_1),\eta(b_1)),\delta(b_1)))\\&=T(\eta(a_1),T(\delta(a_1),T(\eta(b_1),\delta(b_1))))\\&=T(T(\eta(a_1),\delta(a_1)),T(\eta(b_1),\delta(b_1)))\\&=T(T(\eta,\delta)(a_1),T(\eta,\delta)(b_1)).\end{aligned}$$

Therefore $x_1+a_1+z_1=b_1+z_1$ implies

$$T(\eta,\delta)(x_1)\geq T(T(\eta,\delta)(a_1),T(\eta,\delta)(b_1))$$

for $x_1,a_1,b_1,z_1\in S$.

Hence $T(\eta,\delta)$ is a T-fuzzy right h-ideal of S, Similarly, we prove that if η and δ are T-fuzzy left h-ideals of S, then $T(\eta,\delta)$ is a T-fuzzy left h-ideal of S.

Corollary 3.18. [7] *Let η and δ be the fuzzy sets of S. If η and δ are fuzzy right [left] h-ideals of S, then $\eta\cap\delta$ is a fuzzy right [left] h-ideal of S.*

Proof. Instead of T-norm, if we take minimum T-norm in Theorem 3.17, we get the result.

REFERENCES

[1] P. Dheena and G. Mohanraj, "T-fuzzy ideals in rings," International Journalof Computational Cognition, 9(2), pp. 98-101, 2011.

[2] P. Dheena and G. Mohanraj, "(λ,μ)-fuzzy ideals in Semirings," Advances in Fuzzy Mathematics, 6(2), pp. 183-192, 2011.

[3] P. Dheena and G. Mohanraj, "On Intuitionistic Fuzzy K -ideals of Semirings," International Journal of Computational Cognition, 9(2), pp. 45-50, 2011.

[4] P. Dheena and G. Mohanraj, "On (λ,μ)-Fuzzy Prime ideals of Semirings," The Journal of Fuzzy Mathematics, 20(4), pp. 889-898, 2012.

[5] P. Dheena and G. Mohanraj, "Fuzzy Small Ideals of Ring," Journal of Hyperstructures, 2(1), pp. 8-17, 2013.

[6] J.S. Golan, "Semirings and their applications," Kluwer Academic Publishers, Dordrecht, 1999.

[7] Y.B. Jun, M.A. Ozturk and S.Z. Song, "On fuzzy h-ideals in hemirings," Information Science, 162, pp. 211-226, 2004.

[8] E.P. Klement, R. Mesiar and E. Pap, "Triangular Norms," Kluwer Academic Publishers, Dordrecht, 2000.

[9] D.R. La Torre, "On h-ideals and k-ideals in hemirings," Publ. Math. Debrecen, 12, pp. 219-226, 1965.

[10] G. Mohanraj, R. Hema and E. Prabu, "On various weak fuzzy prime ideals of ordered semigroup", Proceedings of the International Conference on Mathematical Sciences published by Elsevier, pp. 475-479, 2014.

[11] G. Mohanraj and E. Prabu, "Weakly fuzzy prime ideal of hemiring," Proceedings of the International Conference on Mathematical Sciences published by Elsevier, pp. 480-483, 2014.

[12] G. Mohanraj and E. Prabu, "Generalized Fuzzy Right h-Ideals of Hemirings," International Journal of Fuzzy Mathematical Archive," 7(2), pp. 147-155, 2015.

[13] R. Rosenfeld, "Fuzzy groups," Journal of Mathematical Analysis and Applications, 35, pp. 512-517, 1971.

[14] L.A. Zadeh, "Fuzzy sets," Information and Control, 8, pp. 338-353, 1965.

[15] J. Zhan, W.A. Dudek, "Fuzzy h-ideals of hemirings," Information Science, 177, pp. 876-886, 2007.

[16] Xueling Ma, Yunqiang Yin and Jianming Zhana, "Characterizations of h-intra- and h-quasi-hemiregular hemirings" Computers and Mathematics with Applications, 63, pp. 783-793, 2012.