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A Note on Explicit Evaluation of Ramanujan's Cubic Continued Fraction using Theta Function Identities

*Yudhisthira Jamudulia**
Department of Mathematics,
Ramanujan School of Mathematical Sciences,
Puducherry-605014, India.
Email: yudhismath@hotmail.com

Syeda Noor Fathima
Department of Mathematics,
Ramanujan School of Mathematical Sciences,
Puducherry-605014, India.
Email: fathima.mat@pondiuni.edu.in

Abstract- **In this paper, we derive some general theorems for the explicit evaluation of Ramanujan's cubic continued fraction employing theta function identities.**

Keywords- Theta Functions, Continued Fraction.

I. INTRODUCTION

The following beautiful continued fraction, communicated by Ramanujan in his second letter to Hardy

$$G(q) := \frac{q^{1/3}}{1}\,\frac{q+q^2}{+\;1+}\,\frac{q^2+q^4}{1\;+}\,\frac{q^3+q^6}{1\;+\cdots},\quad |q|<1 \qquad (1.1)$$

has recorded on page 366 of his lost notebook [12]. H. H. Chan [6] has discovered many new identities which perhaps are the identities to which Ramanujan vaguely referred. Several new modular equation relating G(q) its explicit evaluations over years are given by several mathematicians. We mention here specially B. C. Berndt, Chan and L-C Zhang [4]. For more works on evaluation of the cubic continued fraction one may see [5], [11], [10], [8], [9] and [7].

Motivated by these works in Section 2 of this paper, we establish some new formulas for evaluating G(q) by employing certain identities found in the works of [2, Entry 62, pp.221] and [3, pp. 127]. As a particular case of our general formulas, we deduce certain known numerical values of G (q).

We conclude this introduction with few customary definition we make use in the sequel. For a and q complex number with $|q|<1$

$$(a)_\infty := (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$$

and $(a)_n := (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) = \frac{(a)_\infty}{(aq^n)_\infty}$, n: any integer,

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2},$$

$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \qquad |ab|<1.$$

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^{n(3n-1)/2} = (q;q)_\infty$$

The following identities are quite useful for constructing new theorems ahead

$$e^{-\alpha/24}\sqrt[4]{\alpha}\, f(e^{-\alpha}) = e^{-\beta/24}\sqrt[4]{\beta}\, f(e^{-\beta}),\ \alpha\beta=\pi^2 \qquad (1.2)$$

If G (q) is defined by (1.1), then

$$\left(27+\frac{f^{12}(-q)}{qf^{12}(-q^3)}\right)^{1/3} = \frac{1}{G(q)} + 4G^2(q)$$

Equations (1.2) and (1.3) are found in [1, Ch. 16, Entry 27, pp. 43] and [1, Ch. 20, Entry 1, pp. 345] respectively.

Along with the identities, the following modular equations are also used to find general theorems.

Theorem 1.1. [2]

If $P=\frac{f(-q)}{q^{1/12}(-q^3)}$, $Q=\frac{f(-q^5)}{q^{1/12}(-q^{15})}$, *then*

$$(PQ)^2+5+\frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3.$$

For the proof we refer [2, Entry 62, pp.221].

Theorem 1.2. [3]

If $P=\frac{f(-q)}{q^{1/12}(-q^3)}$, $Q=\frac{f(-q^{11})}{q^{1/12}(-q^{33})}$, *then*

$$(PQ)^5+\left(\frac{3}{PQ}\right)^5+11\left\{(PQ)^4+\left(\frac{3}{PQ}\right)^4\right\}+66\left\{(PQ)^3+\left(\frac{3}{PQ}\right)^3\right\}+253\left\{(PQ)^2+\left(\frac{3}{PQ}\right)^2\right\}+693\left(PQ+\frac{3}{PQ}\right)$$
$$+1386=\left(\frac{Q}{P}\right)^6+\left(\frac{P}{Q}\right)^6.$$

For the proof we refer [3, pp. 127]

2. EVALUATION

In this section we present some theorems for explicit evaluation of G (q).

Theorem 2.1.

For $q=e^{-\pi\sqrt{n/3}}$, let

$$J_n=\frac{f(q)}{3^{1/4}q^{1/12}(q^3)} \qquad (2.1)$$

Then

$$J_nJ_{1/n}=1 \qquad (2.2)$$

$$J_1=1 \qquad (2.3)$$

*Corresponding author.

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Proof. By the definitions of J_n and $J_{1/n}$ as in (2.1) and (1.2), we obtain (2.2). Then setting n=1 in (2.2) we easily obtain (2.3).

Theorem 2.2.

$$3(J_nJ_{25n})^2-5+\frac{3}{(J_nJ_{25n})^2}=\left(\frac{J_{25n}}{J_n}\right)^3-\left(\frac{J_n}{J_{25n}}\right)^3 \qquad (2.4)$$

Proof. Replacing q to –q in Theorem 1.1, we obtain

$$(RS)^2-5+\frac{3}{(RS)^2}=\left(\frac{S}{R}\right)^3-\left(\frac{R}{S}\right)^3 \qquad (2.5)$$

where R= $\frac{f(q)}{q^{1/12}(q^3)}$and S= $\frac{f(q^5)}{q^{5/12}(q^{15})}$. It can be easily seen that R=$3^{1/4}J_n$ and S= $3^{1/4}J_{25n}$. Substituting these R and S in (2.5), we obtain (2.4).

Theorem 2.3.

We have $J_5=\sqrt[6]{\frac{1+\sqrt{5}}{2}}$ and $J_{1/5}=\sqrt[6]{\frac{2}{1+\sqrt{5}}}$.

Proof.

Setting n=1/5 in (2.4) and employing (2.2), we obtain

$$J_5^{12}-J_5^6-1=0.$$

Since $J_n>0$, we obtain by solving the above equation $J_5=\sqrt[6]{\frac{1+\sqrt{5}}{2}}$. Again employing (2.2), we obtain $J_{1/5}=\sqrt[6]{\frac{2}{1+\sqrt{5}}}$.

Theorem 2.4.

We have $J_{25}=\frac{2}{\sqrt{5}-1}$ and $J_{1/25}=\frac{\sqrt{5}-1}{2}$.

Proof. Setting n=1 in (2.4) and observing that $J_1=1$, we have the following quadratic equation $J_{1/25}^2+J_{1/25}-1=0$.
Solving this we obtain $J_{1/25}=\frac{\sqrt{5}-1}{2}$. Using (2.2), we obtain $J_{25}=\frac{2}{\sqrt{5}-1}$.

Theorem 2.5.

$$\left\{\left(3^{1/2}J_nJ_{121n}\right)^5+\left(\frac{3}{3^{1/2}J_nJ_{121n}}\right)^5\right\}-11\left\{\left(3^{1/2}J_nJ_{121n}\right)^4+\left(\frac{3}{3^{1/2}J_nJ_{121n}}\right)^4\right\}+66\left\{\left(3^{1/2}J_nJ_{121n}\right)^3+\left(\frac{3}{3^{1/2}J_nJ_{121n}}\right)^3\right\}-253\left\{\left(3^{1/2}J_nJ_{121n}\right)^2+\left(\frac{3}{3^{1/2}J_nJ_{121n}}\right)^2\right\}+693\left\{3^{1/2}J_nJ_{121n}+\left(\frac{3}{3^{1/2}J_nJ_{121n}}\right)\right\}-1386=\left(\frac{J_{121n}}{J_n}\right)^6+\left(\frac{J_n}{J_{121n}}\right)^6.$$

Proof.

$$\left\{RS^5+\left(\frac{3}{RS}\right)^5\right\}-11\left\{(RS)^4+\left(\frac{3}{RS}\right)^4\right\}+66\left\{(RS)^3+\left(\frac{3}{RS}\right)^3\right\}-253\left\{(RS)^2+\left(\frac{3}{RS}\right)^2\right\}+693\left\{(RS)+\left(\frac{3}{RS}\right)\right\}-1386=\left(\frac{S}{R}\right)^6+\left(\frac{R}{S}\right)^6,$$

where R= $\frac{f(q)}{q^{1/12}(q^3)}$and S= $\frac{f(q^{11})}{q^{11/12}(q^{33})}$. It is easily follows that R= $3^{1/4}J_n$ and S=$3^{1.4}J_{121n}$. Substituting these in (2.7) we obtain (2.6).

Theorem 2.6.

We have$J_{11}=\sqrt[12]{\frac{1+\sqrt{A^2-4}}{2}}andJ_{1/11}=\sqrt[12]{\frac{2}{1+\sqrt{A^2-4}}}$
where A= $1810\sqrt{3}$- 3894.

Proof.

Setting n=1/11 in (2.6) and employing (2.2) we obtain

$$J_{11}^{24}-J_{11}^{12}A+1=0.$$

Solving this equation we obtain $J_{11}=\sqrt[12]{\frac{1+\sqrt{A^2-4}}{2}}$. Again using (2.2) we have $J_{1/11}=\sqrt[12]{\frac{2}{1+\sqrt{A^2-4}}}$.

Theorem 2.7.

$27(1-J_n^{12})=\left(\frac{1}{w}+4w^2\right)^3$ where w= G (-q).

Proof. Replacing q by –q in (1.3) , we have

$$\left(27-\frac{f^{12}(q)}{qf^{12}(q^3)}\right)^{1/3}=\frac{1}{G(-q)}+4G^2(-q).$$

Employing the definition of J_n, we complete the proof of (2.8).

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