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Adjacent vertex distinguishing total coloring of grid graphs and shadow graphs

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***Abstract-* An adjacent vertex distinguishing total coloring of a graph G is a proper coloring of G in such a way that any pair of adjacent vertices have distinct set of colors. The minimum number of colors needed for an adjacent vertex distinguishing total coloring of G is denoted by $\chi_{at}(G)$. In this paper, we have discussed the adjacent vertex distinguishing total coloring of grid of diamonds, grid of hexagons and shadow graphs of (i) a path and (ii) a cycle. We have also discussed the adjacent vertex distinguishing total coloring of a crown graph.**

Keywords- Grid of diamonds; Grid of hexagons; Shadow graph; Path; Cycle; Crown graph; Adjacent vertex distinguishing total coloring and Adjacent vertex distinguishing total chromatic number.

AMS Subject Classification: 05C15

I. INTRODUCTION

If $G=\left(V\left(G\right),E\left(G\right)\right)$ is a graph with the vertex set $V\left(G\right)$ and the edge set $E\left(G\right)$, a proper total coloring of G is an assignment of colors to the vertices and the edges in such a way that

1. no two adjacent vertices are assigned with the same color,
2. no two adjacent edges are assigned with the same color,
3. no edge and its end vertices are assigned with the same color and
4. for every adjacent vertices have distinct set of colors.

Zhang et al. [1] introduced the concept of an adjacent vertex distinguishing total coloring and found the adjacent vertex distinguishing total chromatic number for a cycle, a complete graph, a complete bipartite graph, a wheel and a tree. They have also posed the following conjecture:

For any graph G with order at least 2,
we have $\chi_{at}(G) \leq (G)+3.$

Chen [2] and Wang [3] confirmed that this conjecture is true for graphs with $\Delta(G)=3$ whereas Hulgan [4] presented a proof for this conjecture for a complete graph and a cycle. Chen et al. [5] have verified this conjecture for a generalized Halin graphs with maximum degree at least 6. Wang et al. [6] have verified this conjecture for planar graphs. Papaioannou et al. [7] have discussed adjacent vertex distinguishing total coloring of 4 - regular graphs. Sudha et al. [8] have discussed and found in general the adjacent vertex distinguishing total coloring of corona product of two paths; two cycles; two complete graphs; a path and a cycle; a cycle and a path; a complete graph and a path; a complete graph and a cycle. Luiz et al. [9] have found the adjacent vertex distinguishing total chromatic number of complete equipartite graph of even order $\Delta(G)+2.$

In this paper, we have obtained the adjacent vertex distinguishing total coloring of grid of diamonds, grid of hexagons and shadow graphs of (i) a path and (ii) a cycle. We have also discussed the adjacent vertex distinguishing total coloring of a crown graph.

II. SUDHA GRID OF DIAMONDS

Definition 2.1. Sudha grid of diamonds $S_d\left(m,n\right)$ is an induced subgraph of the tensor product of two

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paths P_m and P_n (both m and n odd, $m \geq 3$ and $n \geq 3$) with the vertex set and the edge set given by

$$V(S_d(m,n)) = \left\{ \begin{array}{r} (u_i, v_j) / \, either \; i \equiv 1(\text{mod } 2) \; and \; j \equiv 0(\text{mod } 2) \\ or \; i \equiv 0(\text{mod } 2) \; and \; j \equiv 1(\text{mod } 2) \end{array} \right\}$$

and

$$E(S_d(m,n)) = \{(u_i, v_j)(u_k, u_l) / u_i u_k \in E(P_m) \, and \, v_j v_l \in E(P_n)\}$$

Illustration 2.2. Consider Sudha grid of diamonds for $m = 5$ and $n = 5$.

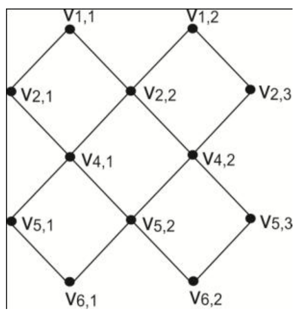

Figure 1: Sudha grid of diamonds $S_d(5,5)$

The above graph is the induced subgraph of the tensor product of two paths P_5 with the vertex set $\{u_i\}$, $1 \leq i \leq 5$ and P_5 with the vertex set $\{v_j\}$, $1 \leq j \leq 5$ as per definition. In the fig.1 the vertices are denoted by $v_{i,j}$ instead of (u_i, v_j) for simplicity.

Theorem 2.3. The adjacent vertex distinguishing total coloring of Sudha grid of diamonds $S_d(m,n)$ is $\Delta(G)+2$ for odd $m \geq 3$ and odd $n \geq 3$.

Proof. Let the vertex set and the edge set of Sudha grid of diamonds $S_d(m,n)$ be given by

$$V(S_d(m,n)) = \{v_{i,j}; 1 \leq \text{i} \leq 2\text{m}+1, 1 \leq \text{j} \leq \text{n}+1\}$$

and

$$E(S_d(m,n)) = \{v_{i,j}v_{i+1,j}; 1 \leq i, j \leq m,n\} \cup \{v_{i,j}v_{i+1,j+1}; 1 \leq i, j \leq m,n\} \cup \{v_{i,j}v_{i+1,j-1}; 1 \leq i, j \leq m,n\}$$

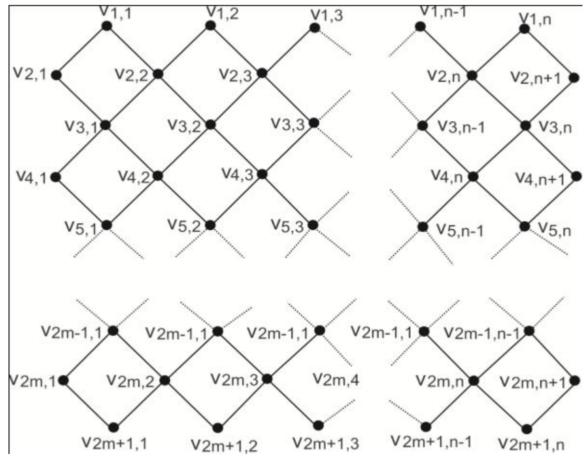

Figure 2: Sudha grid of diamonds $S_d(m,\text{n})$

$S_d(m,n)$ consists of $\left(\frac{mn-1}{2}\right)$ vertices and $(m-1)(n-1)$ edges respectively.

Define the function f_1 be a mapping from the vertices to a color set $\{1,2,3,...,k\}$ as for all $1 \leq i \leq m$, $1 \leq j \leq n$,

$$f_1(v_{i,j}) = \begin{cases} 1, & \text{if} \;\; i \equiv 1(\text{mod } 2) \\ 2, & \text{otherwise} \end{cases}$$

Define the function f_2 be a mapping from the edges to a color set $\{1,2,3,...,k\}$ as for all $1 \leq i \leq m$, $1 \leq j \leq n$,

$$f_2(v_{i,j}v_{i+1,j}) = \begin{cases} 3, & \text{if} \;\; i \equiv 1(\text{mod } 2) \\ 5, & \text{otherwise} \end{cases}$$

and for the remaining edges, there are two cases for f_2 one for odd i and one for even i: for odd i, the function f_2 is defined as

for all $1 \leq i \leq m$, $1 \leq j \leq n$,

$$f_2(v_{i,j}v_{i+1,j+1}) = 4,$$

For even i, the function f_2 is defined as

for all $1 \leq i \leq m$, $1 \leq j \leq n$,

$$f_2(v_{i,j}v_{i+1,j+1}) = 6,$$

The above coloring pattern satisfies the condition of an adjacent vertex distinguishing total

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coloring and the chromatic number of Sudha grid of diamonds $S_d(m,n)$ is $\Delta(G)+2$.

Illustration 2.4. Consider the Sudha grid of diamonds $S_d(7,5)$

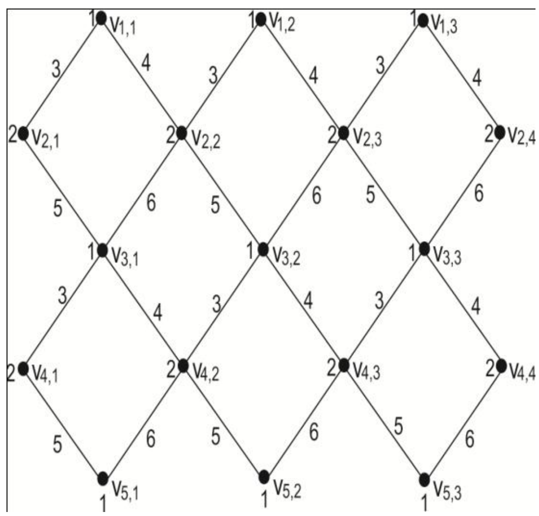

Figure 3: Sudha grid of diamonds $S_d(7,5)$

By using the coloring pattern given in theorem 2.3, the colors $1,2,3,4,5,6$ are assigned to the vertices and the edges are assigned with the colors as shown in fig.3.

The adjacent vertex distinguishing total chromatic number of $S_d(7,5)$ is 6.

III. SUDHA GRID OF HEXAGONS

Definition 3.1. Sudha grid of hexagons $S_h(m,n)$ is an induced subgraph of the strong product of two paths P_m and P_n (m is odd, ≥ 3 and $n \equiv 0(mod 4)$) with the vertex set and the edge set given by

$$V(S_h(m,n)) = \left\{ \begin{array}{c} (u_i, v_j) / i+j \equiv 1(\text{mod } 2) \\ \text{and} \quad i+j \equiv 0(\text{mod } 2) \end{array} \right\}$$

and

$$E(S_h(m,n)) = \left\{ \begin{array}{l} (u_i,v_j)(u_k,v_l) / u_i u_k \in E(P_m) \text{and} \\ v_j v_l \in E(P_n), u_i u_k \in E(P_m) \; and \; j=l \end{array} \right\}$$

Illustration 3.2. Consider the Sudha grid of hexagons $S_h(7,8)$

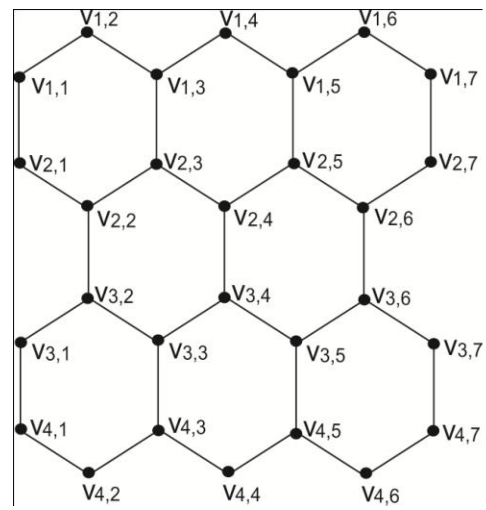

Figure 4: Sudha grid of hexagons $S_h(7,8)$

The graph is the induced subgraph of the strong product of two paths P_7 with the vertex set $\{u_i\}$, $1 \leq i \leq 7$ and P_8 with the vertex set $\{v_j\}$, $1 \leq j \leq 8$.

Theorem 3.3. The adjacent vertex distinguishing total coloring of Sudha grid of hexagons $S_h(m,n)$ is $\Delta(G)+2$ for odd $m \geq 3$ and $n \equiv 0(\text{mod } 4)$.

***Proof*.** Let the vertex set and the edge set of Sudha grid of hexagons $S_h(m,n)$ be given by

$$V(S_h(m,n)) = \{v_{i,j}; 1 \leq \text{i} \leq 2\text{m}+1,\ 1 \leq \text{j} \leq \text{n}+1\}$$

and

$$\text{E}(S_d(m,n)) = \{v_{i,j}v_{i,j+1}, 1 \leq \text{i} \leq 2\text{m}, 1 \leq j \leq n\} \cup \{v_{i,j}v_{i+1,j}, (i+j) \text{ is even}\}$$

$S_h(m,n)$ consists of $\frac{mn}{2}$ vertices and $\frac{mn+2(m-1)(n-1)}{4}$ edges respectively.

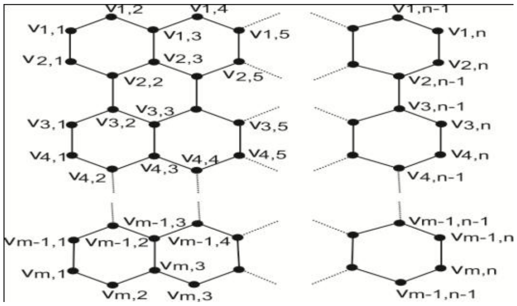

Figure 5: Sudha grid of hexagons $S_h(m, \mathrm{n})$

Define the function f_1 be a mapping from the vertices to a color set $\{1,2,3,...,k\}$ as for all $1 \le i \le m, 1 \le j \le n$,

$$f_1(v_{i,j}) = \begin{cases} 1, & \text{if } (i+j) \equiv 0 \pmod 2 \\ 2, & \text{otherwise} \end{cases}$$

Define the function f_2 to be the mapping from the edges to a color set $\{1,2,3,...,k\}$ as for all $1 \le i \le m, 1 \le j \le n$,

$$f_2(v_{i,j}v_{i,j+1}) = \begin{cases} 3, & \text{if } j \equiv 1 \pmod 2 \\ 4, & \text{otherwise} \end{cases}$$

$$f_2(v_{i,j}v_{i+1,j}) = 5, \text{ if } (i+j) \text{ is even}.$$

The above coloring pattern satisfies the condition of an adjacent vertex distinguishing total coloring and the chromatic number of Sudha grid of diamonds $S_d(m,n)$ is $\Delta(G)+2$.

Illustration 3.4. Consider the Sudha grid of hexagons $S_h(7,8)$

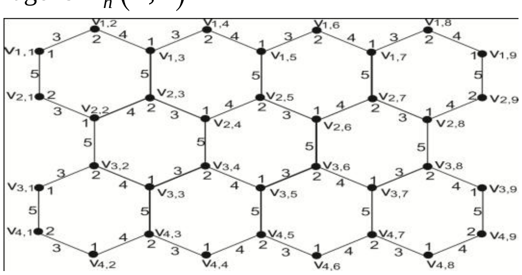

Figure 6: Sudha grid of hexagons $S_h(7,8)$

By using the coloring pattern given in theorem 3.3, the colors $1,2,3,4,5$ to the vertices and the edges are assigned with the colors as shown in fig.6.

The adjacent vertex distinguishing total chromatic number of $S_h(7,8)$ is 5.

IV. CROWN GRAPH

Definition 4.1. The crown graph S_n^0 for an integer $n \ge 2$ is the graph with the vertex set $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and the edge set $\{u_i v_j, 1 \le i, j \le n, \text{ where } i \ne j\}$.

The crown graph S_n^0 has $2n$ vertices and $n(n-1)$ edges.

Illustration 4.2. Consider the crown graph S_4^0.

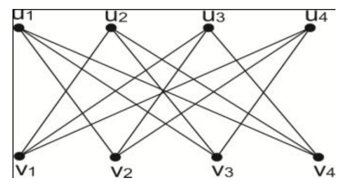

Figure 7: Crown graph S_4^0

The crown graph S_4^0 is shown in fig. 7 with the vertex set $\{u_i\}, 1 \le i \le 4$ and $\{v_j\}, 1 \le j \le 4$.

Theorem 4.3. The adjacent vertex distinguishing total chromatic number of a crown graph S_n^0 is $n+1$ for $n \ge 2$.

Proof. Let the vertex set and the edge set of the crown graph be denoted by

$$V(S_n^0) = \bigcup_{i=1}^{n} \{\{u_i\} \cup \{v_i\}\}$$

and $E(S_n^0) = \bigcup_{i=1}^{n} \{u_i v_j, 1 \le j \le n\}, \text{where } i \ne j$

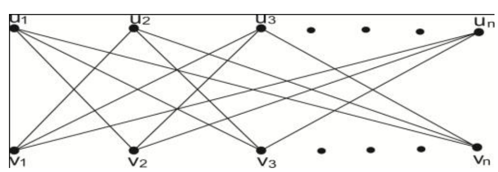

Figure 8: Crown graph S_n^0

Define the functions f_1 and f_2 be the mapping from the vertices and the edges to a color set $\{1,2,3,...,k\}$ as follows:

for all $1 \le i, j \le n,$

$f_1(u_i) = n+1,$

$$f_1(v_j) \equiv \begin{cases} 2j \pmod n, \ if \ 2j \not\equiv 0 \pmod n \\ n, \ \text{otherwise} \end{cases}$$

$$f_2(u_i v_j) \equiv \begin{cases} (i+j) \pmod n, \ if \ (i+j) \not\equiv 0 \pmod n \\ n, \ \text{otherwise} \end{cases}$$

Depending on the nature of n, if the vertices and the edges are colored, the conditions for total coloring is satisfied and we found that the adjacent vertex distinguishing total chromatic number is $n+1$ for $n \ge 2$.

Illustration 4.4. Consider the crown graph $S_4^0 = 5$.

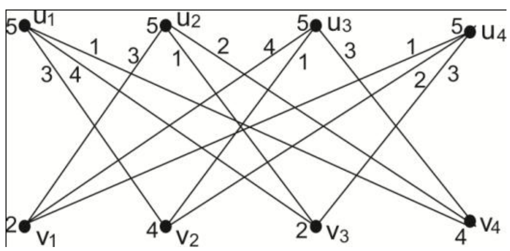

Figure 9: Crown graph S_n^0

The upper vertices denoted by u_1, u_2, u_3 and u_4 are colored with the color 5. The lower vertices v_1, v_2, v_3 and v_4 are colored by the colors $2,4,2$ and 4 respectively. The edges are colored as shown in fig.9. Since only 5 colors are used, the adjacent vertex distinguishing total chromatic number of S_4^0 is 5.

V. SHADOW GRAPHS

Definition 5.1. The Shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex v' in G' to the neighbours of the corresponding vertex v'' in G''.

Illustration 5.2.

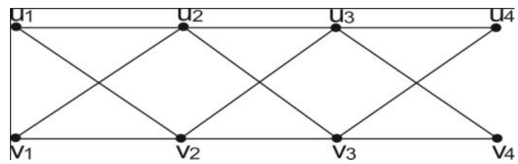

Figure 10: shadow graph of a path $D_2(P_4)$

The shadow graph of the path P_4 is shown in fig.10 with the vertex set $\{u_i\}, 1 \le i \le 4$ and $\{v_j\}, 1 \le j \le 4.$

Theorem 5.3. The adjacent vertex distinguishing total coloring of the shadow graph of a path $D_2(P_n)$ is given by

$$\chi_{at}(D_2(P_n)) = \begin{cases} \Delta(G)+2, & for \quad m > 3 \\ \Delta(G)+1, & for \quad m = 3 \end{cases}$$

Proof. Let the vertex set of the path P_n be $\{v_i, 1 \le i \le n\}.$

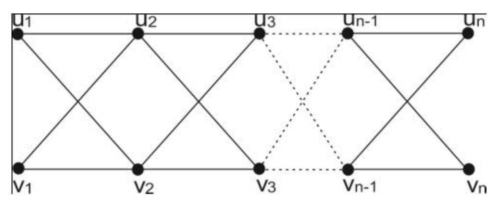

Figure 11: shadow graph of a path $D_2(P_n)$

The vertex set of the shadow graph of the path P_n is

$$V(D_2(P_n)) = \bigcup_{i,j=1}^{n-1} \left\{ \{u_i\} \cup \{v_j\} \right\}$$

and its edge set is

$$E(D_2(P_n)) = \bigcup_{i,j=1}^{n-1} \begin{Bmatrix} \{u_i u_{i+1}\} \cup \{v_j v_{j+1}\} \\ \cup \{u_i v_{j+1}\} \cup \{v_j u_{i+1}\} \end{Bmatrix}$$

Define the functions f_1 and f_2 to be the mapping from the vertices and the edges to a color set $\{1,2,3,...,k\}$ as follows:

for all $1 \le i, j \le n,$

$$f_1(u_i) = \begin{cases} 1, \ if \quad i \equiv 1 \pmod 2 \\ 2, \ \text{otherwise} \end{cases}$$

$$f_1(v_j)=\begin{cases}1, & if\ \ j\equiv 1(\text{mod } 2)\\ 2, & \text{otherwise}\end{cases}$$

$$f_2(u_iu_{i+1})=\begin{cases}3, & if\ \ i\equiv 1(\text{mod } 2)\\ 4, & \text{otherwise}\end{cases}$$

$$f_2(v_jv_{j+1})=\begin{cases}3, & if\ \ j\equiv 1(\text{mod } 2)\\ 4, & \text{otherwise}\end{cases}$$

for all $1\le i, j\le n-1,$

$$f_2(u_iv_{j+1})=5,\ if\ \ i=j$$

$$f_2(v_ju_{i+1})=6,\ if\ \ i=j.$$

Using this general pattern of the coloring, the graph is adjacent vertex distinguishing total colored and the chromatic number is $\Delta(G)+2,$ for $n>3.$

Remark 5.4. When $n=3,$ we found that $\chi_{at}(D_2(P_3))=\Delta(G)+1.$

Illustration 5.5. Consider the shadow graph of a path $D_2(P_4)$.

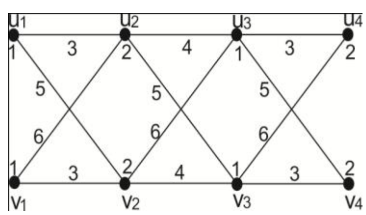

Figure 12: shadow graph of a path $D_2(P_4)$

By using the coloring pattern as given in theorem 5.1, the colors $1,2,3,4,5,6$ are assigned to the vertices and the edges are assigned with the colors as shown in fig.12.
The adjacent vertex distinguishing total chromatic number of $D_2(P_4)$ is 6.

Theorem 5.6. The adjacent vertex distinguishing total coloring of the shadow graph of a cycle $D_2(C_n)$ is $\Delta(G)+2.$

Proof. Let the vertex set of the path C_n be $\{v_i, 1\le i\le n\}.$

The vertex set of the shadow graph of the cycle C_n is

$$V(D_2(C_n))=\bigcup_{i,j=1}^{n}\left\{\{u_i\}\cup\{v_j\}\right\}$$

and its edge set is

$$E(D_2(C_n))=\bigcup_{i,j=1}^{n-1}\left\{\begin{array}{l}\{u_iu_{i+1}\}\cup\{v_jv_{j+1}\}\cup\{u_iv_{j+1}\}\cup\{u_nv_1\}\\ \cup\{v_ju_{i+1}\}\cup\{u_nu_1\}\cup\{v_nv_1\}\cup\{v_nu_1\}\end{array}\right\}$$

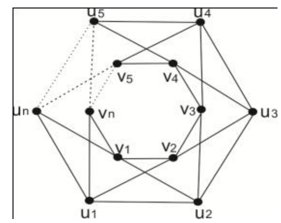

Figure 13: shadow graph of a path $D_2(C_n)$

Define the functions f_1 and f_2 be the mapping from the vertices and the edges to a color set $\{1,2,3,...,k\}$ as follows. There are two cases:

Case 1: Let n be even.

For all $1\le i, j\le n,$

$$f_1(u_i)=\begin{cases}1, & if\ \ i\equiv 1(\text{mod } 2)\\ 2, & \text{otherwise}\end{cases}$$

$$f_1(v_j)=\begin{cases}1, & if\ \ j\equiv 1(\text{mod } 2)\\ 2, & \text{otherwise}\end{cases}$$

for all $1\le i, j\le n-1,$

$$f_2(u_iu_{i+1})=\begin{cases}3, & if\ \ i\equiv 1(\text{mod } 2)\\ 4, & \text{otherwise}\end{cases}$$

$$f_2(v_jv_{j+1})=\begin{cases}3, & if\ \ j\equiv 1(\text{mod } 2)\\ 4, & \text{otherwise}\end{cases}$$

$$f_2(u_nu_1)=3, f_2(v_nv_1)=4$$

for all $1\le i, j\le n-1,$

$$f_2(u_iv_{j+1})=5,\ \ if\ \ i=j$$

$$f_2(v_ju_{i+1})=6,\ if\ \ i=j$$

$$f_2(u_nv_1)=5\ and\ f_2(v_nu_1)=6.$$

Case 2: Let n be odd.

For all $1\le i, j\le n-1,$

$$f_1(u_i)=\begin{cases}1, & if\ \ i\equiv 1(\text{mod } 2)\\ 2, & \text{otherwise}\end{cases}$$

$$f_1(v_j)=\begin{cases}1, & if\ \ j\equiv 1(\text{mod } 2)\\ 2, & \text{otherwise}\end{cases}$$

$$f_1(u_n)=f_1(v_n)=3$$

for all $1 \le i, j \le n-1$,

$$f_2(u_i u_{i+1}) = \begin{cases} 3, \text{ if } i \equiv 1 \pmod 2 \\ 4, \text{ otherwise} \end{cases}$$

$$f_2(v_j v_{j+1}) = \begin{cases} 3, \text{ if } j \equiv 1 \pmod 2 \\ 4, \text{ otherwise} \end{cases}$$

$$f_2(u_n u_1) = f_2(v_n v_1) = 2$$

for all $1 \le i, j \le n-1$,

$$f_2(u_i v_{j+1}) = 5, \text{if } i = j$$

$$f_2(v_j u_{i+1}) = 6, \text{ if } i = j$$

$$f_2(u_n v_1) = 5 \text{ and } f_2(v_n u_1) = 6.$$

Using this general pattern of coloring, the graph $D_2(C_n)$ is adjacent vertex distinguishing total colored and its chromatic number is $\Delta(G)+2,$ for odd n.

Illustration 5.7. Consider the shadow graph of a cycle C_6

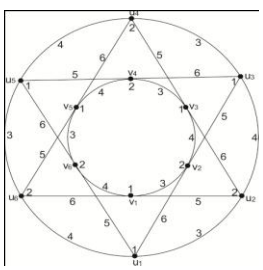

Figure 14: shadow graph of a path $D_2(C_6)$

By using the coloring pattern as given in case 1 of theorem 5.2, the colors $1,2,3,4,5,6$ are assigned to the vertices and the edges are assigned with the colors as shown in fig.14.

The adjacent vertex distinguishing total chromatic number of $D_2(C_6)$ is 6.

Illustration 5.8. Consider the shadow graph of a cycle C_7.

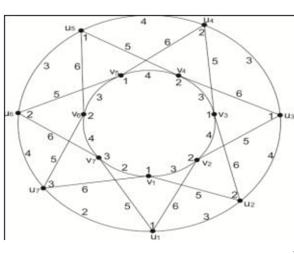

Figure 15: shadow graph of a path $D_2(C_7)$

By using the coloring pattern as given in case 2 of theorem 5.2, the colors $1,2,3,4,5,6$ are assigned to the vertices and the edges are assigned with colors as shown in fig.15.

The adjacent vertex distinguishing total chromatic number of $D_2(C_7)$ is 6.

VI. CONCLUSION

The concept of adjacent vertex distinguishing total chromatic number for the larger graphs obtained from the product of paths are discussed in this paper and found the adjacent vertex distinguishing total chromatic number for Sudha grid of diamonds, Sudha grid of hexagons and shadow graph of (i) a path and (ii) a cycle. We also found the adjacent vertex distinguishing total chromatic number of a crown graph.

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