Global Journal of Pure and Applied Mathematics (GJPAM) ISSN 0973-1768 Volume 12,Number 4 (2016)

On the homogeneous biquadratic equation with 5 unknowns $x^4 - y^4 = 145\,(z^2 - w^2)\,R^2$

P. Jayakumar
Department of Mathematics,
Periyar Maniammai University,
Vallam, Thanjavur - 613403,
Tamil Nadu, India.

*R.Venkatraman**
Department of Mathematics,
Faculty of Engineering and Technology,
SRM University,
Vadapalani, Chennai-600026,
Tamil Nadu, India.
Email: venkarmeeraj@gmail.com

***Abstract*—The Homogenous biquadratic equation with five unknowns given by $x^4-y^4=145(z^2-w^2)\,R^2$ is considered and analyzed for finding its non zero distinct integral solutions. Introducing the linear transformations x = u + v, y = u –v, z =2uv+1, w=2uv-1 and employing the method of factorization different patterns of non-zero distinct integer solutions of the equation under the above equation are obtained. A few interesting relations between the integral solutions and the special numbers namely Polygonal numbers, Pyramidal numbers, Centered Polygonal numbers, Centered Pyramidal numbers, Thabit-ibn-Kurrah number, Star number, Carolnumber, woodall number, kyneanumber, pentatopenumber, stellaoctangula number, octahedral number, Mersenne number are exhibited.**

Keywords*— Homogeneous equation, Integral solutions, Polygonal numbers, Pyramidal numbers and special number.

Notations:

- $T_{m,n}$ - Polygonal number of rank n with size m
- P_n^m - Pyramidal number of rank n with size m
- g_n - Gnomonic number of rank n
- Pr_n - Pronic number of rank n
- $Ct_{16,n}$- Centered hexadecagonal pyramidal number of rank n
- OH_n -Octahedral number of rank n
- SO_n - Stella octangular number of rank n
- ky_n - kynea number
- $carl_n$ -carol number

I. INTRODUCTION

The theory of Diophantine equations offers a richvariety of fascinating problems. In particular biquadratic Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-12]. In this context one may refer [4-10] for various problems on the biquadratic Diophantine equations. However, often we come across non-homogeneous biquadratic equations and as such one may require its integral solution in its most general form. This paper concern with the homogeneous biquadratic equation with five unknowns $x^4-y^4=145(z^2-w^2)\,R^2$ for determining its infinitely many non-zero integral solutions. Also a few interesting properties among the solutions are presented.

*Corresponding author.

This paper was presented by the second author in the National Conference on *Advances in Mathematics and its Applications to Science and Engineering* (AMASE-2016) conducted in Department of Mathematics, University College of Engineering Pattukkottai, Thanjavur, Tamil Nadu, India, on 22nd January 2016.

II. METHOD OF ANALYSIS

The biquadratic equation with five unknowns to be solved for its non-zero distinct integral solution is

$$x^4 - y^4 = 145\,(z^2 - w^2)\,R^2 \qquad (1)$$

Consider the transformations

$$\left.\begin{aligned} x &= u + v \\ y &= u - v \\ z &= 2uv + 1 \\ w &= 2uv - 1 \end{aligned}\right\} \qquad (2)$$

On substituting (2) in (1), we get

$$u^2 + v^2 = 145R^2 \qquad (3)$$

2.1. Pattern I

Assume $145 = (12+i)(12-i)$ (4)

and $R = a^2 + b^2 = (a + ib)(a - ib)$ (5)

Using (4) and (5) in (3) and employing the method of factorization we get

$$(u + iv)(u - iv) = (12+i)(12-i)(a+ib)^2(a-ib)^2$$

On equating the positive and negative factors,

we have,

$$(u + iv) = (12+i)(a+ib)^2$$
$$(u + iv) = (12-i)(a-ib)^2$$

On equating real and imaginary parts, we get

$$u = u(a, b) = 12a^2 - 12b^2 - 2ab$$

$v = v(a, b) = a^2 - b^2 + 24ab$.

On substituting u and v in (2) we get the values of x, y, z and w. The non-zero distinct integrals values of x, y, z, w and R satisfying (1) are given by

$x = x(a, b) = 13a^2 - 13b^2 + 22ab$

$y = y(a, b) = 11a^2 - 11b^2 - 26ab$

$z = z(a, b) = 2(12a^4 + 12b^4 - 72a^2 b^2 + 286a^3b - 286ab^3) + 1$

$w = w(a, b) = 2(12a^4 + 12b^4 - 72a^2 b^2 + 286a^3b - 286ab^3) - 1$

$R = R(a, b) = a^2 + b^2$.

Properties

1. $11x[a, a(2a^2-1)] - 13y[a, a(2a^2-1)] - 580\, SO_a = 0$.
2. $z(a,1) - w(a,1) \equiv 0 \pmod 2$.
3. $11x[(2a-1)^2, 1] - 13y[(2a-1)^2, 1] - 580(G_a)^2 = 0$.
4. $R[(a+1),(a+1)] - 2T_{4,a} - G_{2a} \equiv 0 \pmod 3$.
5. $R(2a, 2a) - 8T_{4,a} = 0$.
6. $x(1,1) - y(1,1) + R(1,1) - P_4^6 = 0$.
7. $11x[a,(2a^2+1)] - 13y[a,(2a^2+1)] - 17400\, OH_a = 0$.
8. $y(a, 2a-1) - ct_{16,a} + 93T_{4,a} - G_{31,a}$ *is a Jacobsthal number*.

2.2. Pattern II

Also 145 can be written in equation (3) as

$$145 = (1 + 12i)(1 - 12i) \quad (6)$$

Using (5) and (6) in equation (3) it is written in factorizable form as $(u + iv)(u - iv) = (1 + 12i)(1 - 12i)(a + ib)^2 (a - ib)^2$

On equating the positive and negative factors, we get,

$$(u + iv) = (1 + 12i)(a + ib)^2$$

$$(u - iv) = (1 - 12i)(a - ib)^2$$

On equating real and imaginary parts, we have

$$u = u(a, b) = a^2 - b^2 - 24ab$$
$$v = v(a, b) = 12a^2 - 12b^2 + 2ab.$$

Substituting the values of u and v in (2), the non-zero distinct values of x, y, z, w and R Satisfying (1) are given by

$x = x(a, b) = 13a^2 - 13b^2 - 22ab$

$y = y(a, b) = -11a^2 + 11b^2 - 26ab$

$z = z(a, b) = 2(12a^4 + 12b^4 - 72a^2 b^2 - 286a^3b + 286ab^3) + 1$

$w = w(a, b) = 2(12a^4 + 12b^4 - 72a^2 b^2 - 286a^3b + 286ab^3) - 1$

$R = R(a, b) = a^2 + b^2$.

Properties

1. $11x[a(a+1), 1] + 13y[a(a+1), 1] + 96P_a = 0$.
2. $R(3,3) - P_3^5 = 0$.
3. $11x(2,a) + 13y(2,a) \equiv 0 \pmod 2$.
4. $x(1,1) - y(1,1)$ *is a Perfect square*.
5. $11x[a, 2a^2+1] + 13y[a, 2a^2+1] - 1740\, OH_a = 0$.
6. $x(a+1, a+1) + y(a+1, a+1) + 48T_{4,a} + G_{43,a} + P_4^6 =$ *Woodall number*.
7. $y(a, a+1) - S_a + 32T_{4,a} - G_a - P_3^6 = 0$.

2.3. Pattern III

Rewrite (3) as

$$1 * u^2 = 145R^2 - v^2 \quad (7)$$

Assume

$$u = 145a - b = (\sqrt{145}\, a + b)(\sqrt{145}\, a - b) \quad (8)$$

Write 1 as,

$$1 = (\sqrt{68} + 1)(\sqrt{68} - 1) \quad (9)$$

Using (8) and (9) in (7) it is written in factorizable from as,

$$(\sqrt{65} + 8)(65 - 8)(\sqrt{65}a + b)^2 (\sqrt{65}a - b)^2 = (\sqrt{65}R + V)(\sqrt{65}R - V) \quad (10)$$

On equating the rational and irrational parts, we get

$$(\sqrt{65} + 8)(\sqrt{65}a + b)^2 = \sqrt{65}R + V$$

$$(\sqrt{65} - 8)(\sqrt{65}a - b)^2 = \sqrt{65}R - V$$

On equating the real and imaginary parts, we get

$$R = R(a, b) = 145a^2 + b^2 + 24ab$$
$$V = v(a, b) = 1740a^2 + 12b^2 + 290ab$$

Substituting the values of u and v in (2), the non–zero distinct integral values of x, y, z, R and w satisfying (1) are given by

$x = x(a, b) = 1885a^2 + 11b^2 + 290ab$

$y = y(a, b) = -1595a^2 - 13b^2 - 290ab$

$z = z(a, b) = 2(252300a^4 - 12b^4 + 42050ab^3 - 290a^3b) + 1$

$w = w(a, b) = 2(252300a^4 - 12b^4 + 42050ab^3 - 290a^3b) - 1$

$R = R(a, b) = 145a^2 + b^2 + 24ab$.

Properties

1. $1595x[(a+1),1]+1885y[(a+1),1]+G_{42050a} \equiv 41 \pmod{2221}$.
2. $R(1,1)+40$ *is a Nasty number*.
3. $w(1,1)+z(1,1) \equiv 0 \pmod 2$
4. $13x[1,a(a+1)]+11y[1,a(a+1)]-580P_a-T_{13,36}$ = *Nasty number*.
5. $x[a(2a-1),1]+y[a(2a-1),1]-290(SOa)+$ *Jacobsthal – lucas number* $=0$.

2.4. Pattern IV

Rewrite (3) as $1 * v^2 = 65R^2 - u^2$ (11)

Write 1 as $1=(\sqrt{65}+1)(\sqrt{65}-1)/64$ (12)

Assume

$v = 65a^2 - b^2 = (\sqrt{65}\,a - b)(\sqrt{65}\,a + b)$ (13)

Using (12) and (13) in (11), it is written in factorizable from as,

$$\frac{(\sqrt{65}+1)(\sqrt{65}-1)}{64}(\sqrt{65}a+b)^2(\sqrt{65}a-b)$$

$$= (\sqrt{65}\,R - u)(\sqrt{65}\,R + u). \quad (14)$$

On equating the rational and irrational factors we get,

$$\left.\begin{aligned} R = R(a,b) &= \frac{1}{8}(65a^2+b^2+2ab) \\ u = u(a,b) &= \frac{1}{8}(65a^2+b^2+130ab) \end{aligned}\right\} \quad (15)$$

Replacing ‘a’ by 8A and ‘b’ by 8B in the above equations (13) and (15), we get

$R = R(A, B) = 1740A^2 + 12B^2 + 24AB$

$u = u(A, B) = 1740A^2 + 12B^2 + 3480AB$

$v = v(A, B) = 20880A^2 - 144B^2$.

On substituting the values of u and v in (2), the non–zero distinct integrals values of x, y, z, w and R satisfying (1) are given by

$x = x(A, B) = 22620A^2 - 132B^2 + 3480AB$

$y = y(A, B) = -19140A^2 + 156B^2 + 3480AB$

$z = z(A, B) = 2(36331200A^4 - 1728B^4 + 72662400A^3B - 501120AB^3) + 1$

$w = w(A, B) = 2(36331200A^4 - 1728B^4 + 72662400A^3B - 501120AB^3) - 1$

$R = R(A, B) = 1740A^2 + 12B^2 + 24AB$.

Properties

1. $\frac{1}{327}.[x(1,1)+y(1,1)] \equiv 0 \pmod 2$.
2. $19140x[(a+1),a]+22620y[(a+1),a]-1002240T_{4,a} \equiv 29 \pmod{5011200}$.
3. $156x[1,a(a+1)]+132[1,a(a+1)]-1002240(P_a) \equiv 29 \pmod{345600}$.
4. $19140x[2a^2+1,a]+22620y[2a^2+1,a]-1002240T_{4,a} - 435974400\,OH_a = 0$.
5. $\frac{1}{768}[x(1,1)-y(1,1)]$ *is a cubic* integer

2.5. Pattern V

Write (3) as $u^2 - R^2 = 64R^2 - v^2$

$(u+R)(u-R) = (8R+v)(8R-v)$ (16)

Which is expressed in the form of ratio as ,

$$\frac{u+R}{8R+v}=\frac{8R-v}{u-R}=\frac{A}{B},\ B\neq 0 \quad (17)$$

This is equivalent to the following two equations,

$-uA + R(8B+A) - VB = 0$

$uB + R(B-8A) - VA = 0$

on solving the above equations by the method of cross multiplication we get,

$u = u(A, B) = -A^2 - B^2 - 16AB$

$R = R(A, B) = -A^2 - B^2$

$v = v(A, B) = 8A^2 - 8B^2 - 2AB$

Substituting the values of u and v in (2), the non – zero distinct integral values of x, y, z, w and R satisfying (1) are given by,

$x = x(A, B) = 11A^2 - 11B^2 - 26AB$

$y = y(A, B) = -13A^2 + 13B^2 - 22AB$

$z = z(A, B) = 2[-12A^4 - 12B^4 + 60A^2B^2 - 286A^3B + 286AB^3] + 1$

$w = w(A, B) = 2[-12A^4 - 12B^4 + 60A^2B^2 - 286A^3B + 286AB^3] - 1$

$R = R(A, B) = -A^2 - B^2$.

Properties

1. $13x[2a^2-1,a]+11y[2a^2-1,a]-580SO_a=0.$
2. $x(a+1,a+2)+26T_{4,a}+G_{50a}\equiv 0 \pmod 2.$
3. $y(a,2a^2-1)-2496DF_a+22SO_a=0.$
4. $13x[a(a+1),1]+11y[a(a+1),1]+580T_{4,a}+G_{290a}$ $= Carol\ number$
5. $x(1,1)+y(1,1)-T_{17,3}=0.$

III. CONCLUSION

It is worth to note that in (2), the transformations for z and w may be considered as z= 2u+v and w =2u –v. for this case, the values of x, y and R are the same as above where as the values of z and w changes for every pattern. To conclude one may consider biquadratic equations with multivariable's ($\geq$5) and search for their non-zero distinct integer solutions along with their corresponding properties.

REFERENCES

[1] Dickson, L.E., History of theory of numbers, Vol.11, Chelsea publishing company, New –York (1952).

[2] Mordell, L.J., Diophantine equation, Academic press, London (1969) Journal of Science and Research, Vol (3) Issue 12, 20-22 (December-14)

[3] Jayakumar. P, Sangeetha, K "Lattice points on the cone $x^2+9y^2=50z^2$" International Journal of Science and Research, Vol (3), Issue 12, 20-22 (December -2014).

[4] Jayakumar P, Kanaga Dhurga, C," On Quadratic Diopphantine equation $x^2+16y^2=20z^2$" Galois J. Maths, 1(1) (2014), 17-23.

[5] Jayakumar. P, Kanaga Dhurga. C, "Lattice points on the cone $x^2+9y^2=50z^2$" Diophantus J. Math, 3(2) (2014), 61-71.

[6] Jayakumar. P, Prabha. S "On Ternary Quadratic Diophantine equation $x^2+15y^2=14z^2$" Archimedes J. Math., 4(3) (2014), 159-164

[7] Jayakumar, P, Meena, J "Integral solutions of the Ternary Quadratic Diophantine equation: $x^2+7y^2=16z^2$" International Journal of Science and Technology, Vol.4, Issue 4, 1-4, Dec 2014.

[8] Jayakumar. P, Shankarakalidoss, G "Lattice points on Homogenous cone $x^2+9y^2=50z^2$" International journal of Science and Research, Vol (4), Issue 1, 2053-2055, January -2015.

[9] Jayakumar. P, Shankarakalidoss. G "Integral points on the Homogenous cone $x^2+y^2=10z^2$" International Journal for Scienctific Research and Development, Vol (2), Issue 11, 234-235, January -2015.

[10] Jayakumar.P, Prapha.S "Integral points on the cone $x^2+25y^2=17z^2$" International Journal of Science and Research Vol(4), Issue 1, 2050-2052, January-2015.

[11] Jayakumar.P, Prabha. S, "Laattice points on the cone $x^2+9y^2=26z2$ "International Journal of Science and Research Vol (4), Issue 1, 2050-2052, January -2015.

[12] Jayakumar. P, Sangeetha. K, "Integral solution of the Homogeneous Biquadratic Diophantine equation with six unknowns: $(x^3-y^3)z=(W^2-P^2)R^4$ "International Journal of Science and Research, Vol(3), Issue 12, 1021-1023 (December-2014).