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Global behavior of certain types of neutral delay difference equations

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***Abstract*— The aim of this paper is to investigate the global behavior of neutral delay difference equations of the first and third order. Examples are provided to prove the results.**

Keywords—Neutral, Delay, Oscillation, Non Oscillation, Solution.

I. INTRODUCTION

First order Neutral Delay Difference Equation is gaining interest because they are the discrete analogue of differential Equations. In recent years, several papers on oscillation of solutions of Neutral Delay Difference Equations have appeared. [2]John R.Graef, R.Savithri, E.Thandapani (John R. Graef, 2004) have analyzed the non oscillatory solutions of first order Neutral Delay Differential Equations with positive coefficient in the Neutral term. [3] Xi-lan Liu and yang-yang dong provided some oscillatory solutions of third order Neutral Delay Differential Equations .

[1]Ozkan Ocalan extensively discusses the problem of Oscillation of neutral differential equation with positive and negative coefficients, J. Math.Anal.Appl. 33(1), 2007, 644-654. [9]Tanaka, S. (2002) discussed the various Solutions of Oscillation First order Neutral Delay Differential Equations.

Here some oscillation results in difference equations based on the existence results of differential equations are provided. Examples are provided to illustrate the results.

II. MAIN RESULTS

Theorem 2.1. If $\sum_{s=N_0}^{n} \rho_s q_s f(1-p_{s-m}) = \infty$ then, every solution of the equation $\Delta(x_n + p_n x_{n-l}) + q_n f(x_{n-m}) = 0,$

is oscillatory, for some $\rho_n, p_n > 0, f_{nm} > f_n f_m, N_0 > 0, l, m > 0.$

Proof. Suppose x_n be a non oscillatory solution of

$$\Delta(x_n + p_n x_{n-l}) + q_n f(x_{n-m}) = 0. \tag{1.1}$$

Without loss of generality let us assume that x_n is eventually positive solution. Let $z_n = x_n + p_n x_{n-l}$. Obviously,

$$\Delta(x_n + p_n x_{n-l}) + q_n f(x_{n-m}) = \Delta z_n + q_n f(z_{n-m} - p_{n-m} x_{n-m-l}) = 0. \tag{1.2}$$

$$\Delta z_n + q_n f(z_{n-m}) f(1-p_{n-m}) < 0 \tag{1.3}$$

Define

$$w_n = \frac{\rho_n \Delta z_n}{f(z_{n-m})} \text{ then } w_n > 0.$$

(1.3) becomes,

$$\frac{w_n f(z_{n-m})}{\rho_n} < -q_n f(z_{n-m}) f(1-p_{n-m})$$

$$w_n < -\rho_n q_n f(1-p_{n-m})$$

Therefore, $\rho_n q_n f(1-p_{n-m}) < -w_n$.

Generlazing, $\sum_{s=N_0}^{n} \rho_s q_s f(1-p_{s-m}) < \infty,$

for $n \geq N_0$. This contradicts the given condition of the theorem. Hence every solution of the equation (1.1) is oscillatory.

Theorem 2.2. Let x_n be a non oscillatory solution of the equation (1.1) and if the following assumptions holds,

$$A_1 : \frac{f(u)}{u} > \gamma > 0, 0 < p_n \leq 1.$$

$$A_2 : Q_s = \sum_{s=N_0}^{n} \gamma q_s (1-p_{s-m}) < \infty,$$

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then $V_{N_0} \geq Q_s + \sum_{s=N_0}^{n} Q_s V_{N_0}$.

Proof.

Let x_n be a non oscillatory solution of the equation (1.1) and without loss of generality, assume that $x_n > 0$. Let $z_n = x_n + p_n x_{n-l}$, $\Delta z_n = -q_n f(z_{n-m} - p_{n-m} x_{n-m-l})$.
From A1 : $\Delta z_n + \gamma q_n f(z_{n-m} - p_{n-m} x_{n-m-l}) \leq 0,$ since $z_n > x_n$,

$$\Delta z_n + \gamma q_n z_{n-m}(q_n - p_{n-m}) \leq 0, \quad (1.4)$$

Define $v_n = \dfrac{z_n}{z_{n-m}}$, $\Delta v_n = \dfrac{z_{n-m}\Delta z_n - z_n \Delta z_{n-m}}{z_{n-m} z_{n-m+1}}$

$$\frac{\Delta v_n z_{n-m} z_{n-m+1} + z_n \Delta z_{n-m}}{z_{n-m}} = \Delta z_n$$

From(1.4)

$$\frac{\Delta v_n z_{n-m} z_{n-m+1} + z_n \Delta z_{n-m}}{z_{n-m}} = -\gamma q_n z_{n-m}(1 - p_{n-m})$$

$$\Delta v_n z_{n-m+1} + \frac{z_n \Delta z_{n-m}}{z_{n-m}} = -\gamma q_n z_{n-m}(1 - p_{n-m})$$

$$\Delta v_n \leq -\gamma q_n z_{n-m}(1 - p_{n-m}).$$

$$\Delta v_n \leq -\gamma q_n z_{n-m}(1 - p_{n-m}) - v_n^{\ 2}$$

taking summation from s = N to n,

$$\sum_{s=N_0}^{n} v_{s+1} - v_s \leq -\sum_{s=N_0}^{n} \gamma q_s (1 - p_{s-m}) - \sum_{s=N_0}^{n} v_s^{\ 2}$$

$$v_{n+1} - v_{N_0} \leq -Q_s - \sum_{s=N_0}^{n} v_s^{\ 2}$$

$$-v_{N_0} \leq -Q_s - \sum_{s=N_0}^{n} v_s^{\ 2} \quad \text{Since } v_{N_0} \geq Q_s \ \_\_$$

$$v_{N_0} \geq Q_s + \sum_{s=N_0}^{n} Q_s v_{N_0}.$$

This completes the theorem.

Example 2.3. Consider the first order neutral delay difference equation $\Delta(x_n + n^2 x_{n-1}) + (2n^2 + 2n - 1) x_{n-1}^{\ 3} = 0$ (1.5)

Equation (1.5) satisfies all conditions of theorem 1.1 and hence all its solutions are oscillatory. One such solution is $x_n = (-1)^n$.

Example 2.4. Consider the first order neutral delay difference equation,

$$\Delta(x_n + \frac{1}{n-2} x_{n-1}) + (\frac{2n^2 - 8n + 7}{(n-1)(n-2)}) x_{n-1}^{\ 2} = 0, n > 1,2 \quad (1.6)$$

Equation (1.6) satisfies all conditions of theorem 1.1 and hence all its solutions are oscillatory. One such solution is $x_n = (-1)^n$.

Theorem 2.5. If y_n is an eventually positive solution of equation $\Delta(a_n \Delta b_n(\Delta(y_n + p_n y_{n-k}))) + q_n f(y_{n-l}) = 0$ (1.7)
and $z_n = y_n + p_n y_{n-k}$ then for sufficiently large n, the following condition exists.

$$z_n > 0, \Delta z_n > 0, \Delta(b_n \Delta z_n) > 0.$$

Proof. Let y_n be an eventually positive solution of equation (1.7). Then there exists $n_1 \geq n_0$ such that $y_{n-k} > 0$, $y_{n-l} > 0$ for $n \geq n_1$.

From the definition of z_n, it is clear that $z_n > 0$ and $\Delta(a_n \Delta b_n \Delta z_n) \leq 0, n \geq n_1$.

We claim that

$$\Delta(b_n \Delta z_n) > 0, \text{ for } n \geq n_2.$$

Suppose $\Delta(b_n \Delta z_n) \leq 0., forn \geq n_2$.

Since $a_n > 0,$ we claim $a_n \Delta(b_n \Delta z_n) < 0,$ for $n \geq n_3$.

Then for $a_n \Delta(b_n \Delta z_n) < 0,$ for $n \geq n_3,$ we get

$$a_n \Delta b_n \Delta z_n \leq a_{n_3} \Delta b_{n_3} \Delta z_{n_3} \leq 0. \quad (1.8)$$

Dividing (1.8) by a_n, we get

$$\Delta b_n \Delta z_n \leq (a_{n_3} \Delta b_{n_3} \Delta z_{n_3}) \frac{1}{a_n} \leq 0.$$

Taking summation from n_3 to n, we obtain

$$\sum_{n_3}^{n}(b_{n+1}-b_n)\Delta z_n \le a_{n_3}\Delta(b_{n_3}\Delta z_{n_3})\sum_{n_3}^{n}\frac{1}{a_n}$$

$$(b_{n_3+1}-b_{n_3})\Delta z_{n_3}+(b_{n_3+2}-b_{n_{33}+1})\Delta z_{n_3+1}+$$

$$(b_{n_3+3}-b_{n_3+2})\Delta z_{n_3+2}+.....+(b_{n+1}-b_n)\Delta z_n \le$$

$$a_{n_3}\Delta(b_{n_3}\Delta z_{n_3})(\frac{1}{a_{n_3}}+\frac{1}{a_{n_3+1}}+.....\frac{1}{a_n})$$

We have, $b_n\Delta z_n \to -\infty, \;\; n\to\infty$.

Hence there exists $n_4 \ge n_3$, such that

$$b_n\Delta z_n \le b_{n_4} < 0, \qquad (1.9)$$

for $n \ge n_4$.

Dividing equation (1.9) by b_n , $\Delta z_n \le b_{n_4}\Delta z_{n_4}\frac{1}{b_n}$.

Taking summation from n_4 to n,

$$\sum_{n_4}^{n}(z_{n+1} - z_n) \le b_{n_4}\Delta z_{n_4}\frac{1}{b_n}$$

$$z_n \to -\infty, n\to\infty.$$

which contradicts the statement of theorem 2.5.

Hence we have $\Delta(b_n\Delta z_n) > 0$ for all n.

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